

# Spherically Averaging Ellipsoidal Galaxy Clusters in X-Ray and Sunyaev-Zel’dovich Studies: I. Analytical Relations

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## ABSTRACT

This is the first of two papers investigating the deprojection and spherical averaging of ellipsoidal galaxy clusters. We specifically consider applications to hydrostatic X-ray and Sunyaev-Zel’dovich (SZ) studies, though many of the results also apply to isotropic dispersion-supported stellar dynamical systems. Here we present analytical formulas for galaxy clusters described by a gravitational potential that is a triaxial ellipsoid of constant shape and orientation. For this model type we show that the mass bias due to spherically averaging X-ray observations is independent of the temperature profile, and for the special case of a scale-free logarithmic potential, there is exactly zero mass bias for any shape, orientation, and temperature profile. The ratio of spherically averaged intracluster medium (ICM) pressures obtained from SZ and X-ray measurements depends only on the ICM intrinsic shape, projection orientation, and  $H_0$ , which provides another illustration of how cluster geometry can be recovered through a combination of X-ray and SZ measurements. We also demonstrate that  $Y_{\text{SZ}}$  and  $Y_{\text{X}}$  have different biases owing to spherical averaging, which leads to an offset in the spherically averaged  $Y_{\text{SZ}} - Y_{\text{X}}$  relation. A potentially useful application of the analytical formulas presented is to assess the error range of an observable (e.g., mass,  $Y_{\text{SZ}}$ ) accounting for deviations from assumed spherical symmetry, without having to perform the ellipsoidal deprojection explicitly. Finally, for dedicated ellipsoidal studies, we also generalize the spherical onion peeling method to the triaxial case for a given shape and orientation.

**Key words:** X-rays: galaxies: clusters — X-rays: galaxies — dark matter — cosmological parameters — cosmology:observations

## 1 INTRODUCTION

Although it is well-known that galaxy clusters are not spherical, spherically averaged measurements of cluster properties are standard practice owing to their comparative simplicity and the expectation that non-spherical effects do not dominate the error budget. However, as cosmological measurements with clusters become ever more precise (e.g., Vikhlinin et al. 2009; Allen et al. 2011; Pierre et al. 2011), the need for precise control of systematic errors also increases. Deviations from spherical symmetry, such as intrinsic flattening and substructure, will necessarily introduce scatter, and possibly biases, into spherically averaged global scaling relations used in cosmological studies (e.g., Evrard et al. 1996; White et al. 2002; Kravtsov et al. 2006; Shaw et al. 2008; Krause et al. 2011); e.g., the scaling between total mass and average intracluster medium (ICM) temperature, quantities that are usually computed interior to a spherical volume specified by a standard fraction of the virial radius.

Unfortunately, owing to their increased complexity and computational expense, non-spherical models are rarely accounted for in detail, if at all, in the error budgets of cluster measurements. The modeling of a cluster possessing substructure requires sophisticated three-dimensional N-body simulations, while even to compute the gravitational potential of simple ellipsoidal mass distributions involves solving complicated integrals (e.g., Chandrasekhar 1987; Binney & Tremaine 2008)

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unless approximate fitting formulas appropriate for nearly spherical systems are employed (Lee & Suto 2004). The pioneering studies by Piffaretti et al. (2003) and Gavazzi (2005) dedicated to the problem of assessing errors arising from the assumption of spherical symmetry employed ellipsoidal models with some simplifying assumptions to reduce the total computational expense; e.g., Gavazzi (2005) studied the face-on projections of spheroidal NFW (Navarro et al. 1997) mass distributions with isothermal ICM. These authors found that quantities obtained from spherical X-ray studies vary typically by  $\lesssim 5\%$  as a result of different intrinsic shapes and viewing orientations for a cluster. While not a large effect in absolute terms, errors of a few percent may be important for precision cosmological studies and deserve further elucidation.

The non-spherical shapes of clusters need not be considered merely a nuisance as a source of systematic error, since they are interesting to study in their own right. With the aid of large, cosmological N-body simulations significant theoretical progress has been made in the understanding of the intrinsic shapes of  $\Lambda$ CDM cluster dark matter halos (e.g., Jing et al. 1995; Mohr et al. 1995; Splinter et al. 1997; Thomas et al. 1998; Jing & Suto 2002; Bullock 2002; Suwa et al. 2003; Springel et al. 2004; Hopkins et al. 2005; Kasun & Evrard 2005; Lee et al. 2005; Allgood et al. 2006; Lee 2006; Paz et al. 2006; Ho et al. 2006; Shaw et al. 2006; Bett et al. 2007; Gottlöber & Yepes 2007; Macciò et al. 2008; Muñoz-Cuartas et al. 2011; Rossi et al. 2011). The axial ratios of dark matter halos in  $\Lambda$ CDM are found to be sensitive to the value of  $\sigma_8$ , the normalization of the power spectrum of density fluctuations, and are weakly sensitive to the cosmological matter density parameter,  $\Omega_m$ . Dark matter halos in  $\Lambda$ CDM become flatter with increasing mass, and cluster-mass halos tend to be prolate/triaxial. Furthermore, halos will be more spherical if the dark matter particle is self-interacting, which provides an additional constraint on particle dark matter models, especially on the galaxy scale (e.g., Feng et al. 2009, 2010).

For a few clusters X-ray observations have measured flattened dark matter halos by adopting hydrostatic models of the ICM (Fabricant et al. 1984; Buote & Canizares 1992, 1996b), and there is recent evidence for triaxiality in the dark matter when X-ray data are combined with constraints from strong gravitational lensing (Morandi et al. 2010, 2011; Morandi & Limousin 2011). Isothermal hydrostatic models of the ICM applied to the dark matter halos formed in cosmological simulations predict distributions of average X-ray isophotal axial ratios consistent with cluster observations (Wang & Fan 2004; Flores et al. 2007; Kawahara 2010). When baryons are included in N-body, hydrodynamical simulations of cosmic structure, it is found that baryon condensation leads to rounder dark matter halos (e.g., Dubinski 1994; Kazantzidis et al. 2004; Debattista et al. 2008). However, the treatment of baryon evolution in cosmological simulations remains a problem for cluster cores. For example, baryon cooling in some cosmological simulations can lead to highly flattened ICM cluster cores that disagree with X-ray observations of relaxed clusters (Fang et al. 2009), whereas the global ICM shapes of the model clusters agree with X-ray observations and also with the shapes of the total gravitational potentials of the models as expected for hydrostatic equilibrium (Buote & Tsai 1995; Fang et al. 2009; Lau et al. 2011). ICM shapes have also been measured via the thermal Sunyaev-Zel'dovich (SZ) effect (De Filippis et al. 2005; Sayers et al. 2011) which provides another promising avenue for probing intrinsic cluster shapes.

The non-spherical results cited above represent only a small minority of X-ray and SZ cluster studies. To a large extent spherical models dominate because they are easier to implement and are substantially more computationally efficient than are models which involve a gravitational potential generated by an ellipsoidal mass distribution. Here we investigate a different type of model where the potential, rather than the underlying mass distribution, is an ellipsoid of constant shape and orientation. These ellipsoidal models lend themselves to straightforward generalizations of simple, analytic spherical models, and are therefore just as computationally efficient. Moreover, we show that many cluster quantities derived assuming spherical symmetry can be easily interpreted in terms of these ellipsoidal models for a given shape and orientation.

The paper is organized as follows. In §2 we define the ellipsoidal models. We show in §3 that the relationship between the mass profile and potential for these models closely resembles the spherical case, as does that between the mass profile and the ICM density and temperature for the case of hydrostatic equilibrium. In §4 we provide several analytic expressions for quantities associated with these ellipsoidal models, in particular relating their deprojected spherical averages to their intrinsic ellipsoidal profiles. For practical use of these models with binned data, in §5 we generalize the traditional spherical onion peeling deprojection method appropriate for a series of concentric, triaxial ellipsoidal shells. Our conclusions are presented in §6. In Paper 2 (Buote & Humphrey 2011b) we perform a detailed investigation of biases and scatter in the measurements of global quantities resulting from the spherical averaging of ellipsoidal galaxy clusters. Finally, we mention that the formulas we present here (and the results in Paper 2) apply not just for massive clusters but also for groups and massive elliptical galaxies with hot gaseous halos.

## 2 ELLIPSOIDAL POTENTIALS

Consider an ellipsoid with principal axes  $a, b, c$  and axis ratios,  $p_v \equiv b/a$  and  $q_v \equiv c/a$ , satisfying  $0 < q_v \leq p_v \leq 1$ . When the ellipsoid is aligned so that  $a$  lies along the  $x$ -axis,  $b$  along the  $y$ -axis, and  $c$  along the  $z$ -axis, then the ellipsoidal radius  $a_v$  is given by,

$$a_v^2 = x^2 + \frac{y^2}{p_v^2} + \frac{z^2}{q_v^2}. \quad (1)$$

We define an ellipsoidal gravitational potential (EP) to be an ellipsoid of constant shape  $(p_v, q_v)$  and orientation so that  $\Phi$  depends only on  $a_v$ ; i.e.,  $\Phi = \Phi(a_v)$ . Since  $\Phi$  has a constant shape, EPs best approximate those clusters having nearly constant ICM shapes. While it is well-known that individual clusters, both those observed and formed in cosmological simulations, can sometimes exhibit large radial shape variations, we note that the average X-ray ellipticity profile for a low-redshift cluster sample varies only weakly with radius (see Fig. 8 of Fang et al. 2009). In contrast, the shapes of the isodensity surfaces of the underlying mass distribution of an EP generally vary with radius. This can be a desirable feature since dark matter halos formed in cosmological simulations typically have radially varying shapes (e.g., Jing & Suto 2002; Kazantzidis et al. 2004; Bailin & Steinmetz 2005; Allgood et al. 2006; Vera-Ciro et al. 2011).

Probably the key advantage of an EP is that a simple, analytic form for  $\Phi$  can be adopted based on a straightforward generalization of a spherical potential, enabling much faster computational evaluation than for ellipsoidal mass distributions; e.g., see Paper 2 for EP generalizations of the NFW and isothermal  $\beta$  models. A disadvantage of an EP is that when the potential is sufficiently flattened (the amount depending on how steep is the radial potential profile), the mass density can become negative in some region (e.g., Binney 1981; Blandford & Kochanek 1987; Schneider et al. 1992; Kassiola & Kovner 1993; Evans 1994; Binney & Tremaine 2008). Requiring a non-negative phase-space distribution function (DF) will further restrict the flattening of the matter distribution that generates the potential if the DF depends only on the energy and one-component of the angular momentum (Evans 1993, 1994), although it is unclear that the same restrictions would apply to a general three-integral DF. Moreover, we show below (§3.1) that the mass enclosed within  $a_v$  has a form analogous to the mass enclosed within the spherical radius  $r$ . It is therefore as well-behaved as the mass profile of a spherical model, indicating that EPs should be suitable for many applications. Indeed, in our previous X-ray studies of elliptical galaxies (Buote & Canizares 1996a; Buote et al. 2002) and a simulated cluster (Buote & Tsai 1995) we found very good agreement between the gravitating mass profiles inferred using EPs with those obtained using the far more computationally expensive ellipsoidal mass distributions.

### 3 MASS DISTRIBUTION

#### 3.1 General Case

As noted previously, unlike  $\Phi$  itself, the mass density  $\rho(x, y, z)$  of an EP is not an ellipsoid, and its typically complicated form must be inferred from direct solution of Poisson's Equation. However,  $M(< a_v)$ , the mass enclosed within  $a_v$ , is far simpler to compute given  $\Phi$ , as we now show.

Gauss's Law for the mass enclosed within a surface  $S$  is,

$$M_{\text{enc}} = \frac{1}{4\pi G} \oint_S \nabla \Phi \cdot \hat{n} dS. \quad (2)$$

Now assuming  $\Phi = \Phi(a_v)$  for an EP and taking  $S$  to be the ellipsoidal surface defined by  $a_v$ , the gradient and vector surface element take the form,

$$\nabla \Phi(a_v) = \left( x\hat{i} + \frac{y}{p_v^2}\hat{j} + \frac{z}{q_v^2}\hat{k} \right) \frac{1}{a_v} \frac{d\Phi}{da_v}, \quad \text{and} \quad \hat{n} dS = \left( \frac{xq_v\hat{i}}{\sqrt{a_v^2 - x^2 - y^2/p_v^2}} + \frac{yq_v\hat{j}}{p_v^2\sqrt{a_v^2 - x^2 - y^2/p_v^2}} + \hat{k} \right) dx dy. \quad (3)$$

Substituting these expressions into eqn. (2) and rearranging into two separate surface integrals yields,

$$M(< a_v) = \frac{1}{4\pi G} \frac{q_v}{a_v} \frac{d\Phi}{da_v} \int \int \frac{x^2 (1 - 1/q_v^2) + a_v^2/q_v^2}{\sqrt{a_v^2 - x^2 - y^2/p_v^2}} dx dy + \frac{1}{4\pi G} \frac{q_v}{a_v} \frac{d\Phi}{da_v} \int \int \frac{y^2}{p_v^2} \left( \frac{1}{p_v^2} - \frac{1}{q_v^2} \right) \frac{1}{\sqrt{a_v^2 - x^2 - y^2/p_v^2}} dx dy. \quad (4)$$

Because of the ellipsoidal symmetry we may evaluate the integrals in any quadrant of the  $x - y$  plane. Multiplying the result obtained from one quadrant by four gives the total surface integral in the positive  $z$ -direction, and multiplying by eight gives the result for the total ellipsoidal surface. By effecting a change of variable,  $y = up_v\sqrt{a_v^2 - x^2}$ , the first term on the R.H.S. in eqn. (4) simplifies to,

$$\frac{2}{\pi G} \frac{p_v q_v}{a_v} \frac{d\Phi}{da_v} \int_0^{a_v} dx \left[ x^2 \left( 1 - \frac{1}{q_v^2} \right) + \frac{a_v^2}{q_v^2} \right] \int_0^1 \frac{du}{\sqrt{1 - u^2}} = \frac{a_v^2}{G} \frac{d\Phi}{da_v} \frac{p_v q_v}{3} \left( 1 + \frac{2}{q_v^2} \right). \quad (5)$$

Here the  $u$ -integral evaluates to  $\pi/2$  so that the  $x$ -integral is elementary. Similarly, making the same change of variable in the second term on the R.H.S. in eqn. (4) yields,

$$\frac{2}{\pi G} \frac{p_v q_v}{a_v} \frac{d\Phi}{da_v} \left( \frac{1}{p_v^2} - \frac{1}{q_v^2} \right) \int_0^{a_v} (a_v^2 - x^2) dx \int_0^1 \frac{u^2 du}{\sqrt{1 - u^2}} = \frac{a_v^2}{G} \frac{d\Phi}{da_v} \frac{p_v q_v}{3} \left( \frac{1}{p_v^2} - \frac{1}{q_v^2} \right), \quad (6)$$

where this time the  $u$ -integral evaluates to  $\pi/4$ , again leading to an elementary  $x$ -integral. Substituting the results from equations (5) and (6) back into eqn. (4) gives the result for the enclosed mass, which we state as a theorem:

**Theorem 1.** For an EP the mass enclosed within the ellipsoid of radius  $a_v$  is,

$$M(< a_v) = \eta(p_v, q_v) \frac{a_v^2}{G} \frac{d\Phi}{da_v}, \quad (7)$$

where we define the EP shape factor,

$$\eta(p_v, q_v) \equiv \frac{p_v q_v}{3} \left[ 1 + \frac{1}{p_v^2} + \frac{1}{q_v^2} \right], \quad (8)$$

which is of order unity.

Hence, the relation between  $M(< a_v)$  and  $\Phi(a_v)$  is nearly identical in form to the spherical case,  $M(< r) = (r^2/G)d\Phi/dr$ , to which it reduces for  $p_v = q_v = 1$ . Here it is worth emphasizing that the axis ratios  $p_v$  and  $q_v$  define the shape of  $\Phi$  and the bounding surface for the calculation of  $M(< a_v)$ , but generally the axis ratios of the corresponding mass density distribution are smaller (e.g., see Fig. 1 of Paper 2).

The spherically averaged mass distribution of an EP does not possess such a simple relationship to the potential. Nevertheless, it can be computed without resorting to the evaluation of the volume integral of the generally complicated density distribution.

**Theorem 2.** For an EP the mass enclosed within the sphere of radius  $r$  is,

$$M(< r) = \frac{r}{4\pi G} \int_{4\pi} a_v \frac{d\Phi(a_v)}{da_v} d\Omega = \frac{r}{4\pi \eta(p_v, q_v)} \int_{4\pi} \frac{M(< a_v)}{a_v} d\Omega, \quad (9)$$

where  $d\Omega = \sin \theta d\theta d\phi$  is the solid angle and the integration proceeds over the entire spherical surface.

*Proof.* Again using Gauss's Law (eqn. 2), but this time for a spherical surface, gives,

$$\hat{n} dS = \hat{r} r^2 d\Omega = (x\hat{i} + y\hat{j} + z\hat{k}) r d\Omega, \quad (10)$$

for the vector surface element. Taking the dot product with  $\nabla\Phi(a_v)$  (eqn. 3) and integrating over the sphere gives the stated result, where the R.H.S. made use of Theorem 1.  $\square$

### 3.2 Hydrostatic Equilibrium

Since a primary goal of our study is to investigate the effect of spherical averaging on the inferred mass distribution from X-ray observations, we need to consider the case where the hot intracluster medium (ICM), or “hot gas”, is in hydrostatic equilibrium. We assume the self-gravity of the gas can be neglected ( $\Phi_{\text{gas}} \ll \Phi$ ) which is generally a very good approximation interior to  $r_{500}$  where the cluster gas fraction is  $\sim 10\%$  (e.g., Pratt et al. 2010). In this case, the equation of hydrostatic equilibrium,

$$\nabla P_{\text{gas}} = -\rho_{\text{gas}} \nabla \Phi, \quad (11)$$

where  $P_{\text{gas}}$  is the thermal pressure and  $\rho_{\text{gas}}$  is the density of the ICM, requires that surfaces of constant potential are the same as surfaces of constant ICM pressure, density, temperature, and, so long as the metal abundances do not vary over these surfaces, it follows that the surfaces of constant X-ray emissivity also follow the potential (“X-ray Shape Theorem,” Buote & Canizares 1994, 1996a; Buote & Humphrey 2011a). For the special case of an EP model it follows that all ICM quantities depend only on  $a_v$ ; e.g.,  $P_{\text{gas}} = P_{\text{gas}}(a_v)$ ,  $\rho_{\text{gas}} = \rho_{\text{gas}}(a_v)$ , and  $T = T(a_v)$ , so that the hydrostatic equation becomes,

$$\frac{dP_{\text{gas}}(a_v)}{da_v} = -\rho_{\text{gas}}(a_v) \frac{d\Phi(a_v)}{da_v}, \quad (12)$$

where we have used the definition of the gradient in eqn. (3). Using Theorem 1 and substituting the ideal gas equation of state for the pressure,  $P_{\text{gas}} = \rho_{\text{gas}} k_B T / (\mu m_a)$ , where  $k_B$  is Boltzmann's constant,  $m_a$  is the atomic mass unit, and  $\mu$  is the mean atomic weight of the gas, we obtain the following result.

**Theorem 3.** For an EP in hydrostatic equilibrium the mass enclosed within an ellipsoid of radius  $a_v$  is,

$$M(< a_v) = -\eta(p_v, q_v) \left[ \frac{a_v k_B T}{\mu m_a G} \right] \left[ \frac{d \ln \rho_{\text{gas}}}{d \ln a_v} + \frac{d \ln T}{d \ln a_v} \right], \quad (13)$$

where  $\eta$  is given by eqn. (8).

This result is a simple generalization of the spherical case where  $r$  is replaced by  $a_v$  and the mass is multiplied by the shape factor  $\eta(p_v, q_v)$ . Similarly, solutions of eqn. (12) for  $\rho_{\text{gas}}$ ,  $T$ , and the entropy are also easily constructed from their spherical counterparts by making the transformation  $r \rightarrow a_v$  and  $M(< r) \rightarrow M(< a_v)/\eta$  (e.g., see equations 10-12 of Buote & Humphrey 2011a).

## 4 DEPROJECTION RELATIONS FOR SPHERICALLY AVERAGED QUANTITIES

Here we derive analytical expressions for the spherically averaged deprojection of intrinsically ellipsoidal quantities, with a particular application to the EPs. We begin by summarizing the general results for the projection of a triaxial ellipsoid.

### 4.1 Preliminaries

We define the orientation of the ellipsoidal system following Binney (1985). In the reference  $(x, y, z)$  coordinate system the principal axes of the ellipsoid are aligned with the coordinate directions as described in the definition of  $a_v$  (eqn. 1). Define a new rotated coordinate system  $(x', y', z')$  where the  $z'$  axis lies along the line-of-sight to the observer from the center of the ellipsoid and the  $x'$  axis is located in the  $(x, y)$  plane. The two systems are related by starting with their axes aligned, rotating the reference system first by an angle  $\phi$  about the  $z'$  axis and then by an angle  $\theta$  about the  $x'$  axis. (For the rotation matrix, see Binney 1985.)

For any quantity that depends only on the ellipsoidal radius  $a_v$ , Contopoulos (1956) and Stark (1977) showed that the projection of this ellipsoidal quantity yields a (two-dimensional) elliptical distribution of constant shape and orientation that depends only on the elliptical coordinate  $a_s$  on the sky; i.e., the projection of the ellipsoidal volume emissivity  $\epsilon(a_v)$  (i.e., luminosity per volume) yields an elliptical surface brightness  $\Sigma(a_s)$ ,

$$\Sigma(a_s) = \frac{2}{\sqrt{f}} \int_{a_s}^{\infty} \frac{\epsilon(a_v) a_v da_v}{\sqrt{a_v^2 - a_s^2}}, \quad (14)$$

where,

$$f = \sin^2 \theta \left( \cos^2 \phi + \frac{\sin^2 \phi}{p_v^2} \right) + \frac{\cos^2 \theta}{q_v^2}, \quad (15)$$

using the angle definitions described above, and where  $p_v$  and  $q_v$  define  $a_v$ . The elliptical coordinate variable  $a_s = \gamma_s q_s \alpha$  is proportional to the elliptical radius  $\alpha$  (i.e., semi-major axis on the sky), which is of more immediate interest to the observer,

$$\alpha^2 = X^2 + \frac{Y^2}{q_s^2}, \quad (16)$$

where  $q_s$  is the axial ratio of  $\Sigma$ , and  $X$  and  $Y$  are sky coordinates aligned with the isophotal major and minor axes respectively<sup>1</sup>. The proportionality factors are given by,

$$\gamma_s^2 = \frac{1}{2f} \left[ (A + C) + \sqrt{(A - C)^2 + B^2} \right], \quad (17)$$

$$q_s^2 = \frac{(A + C) - \sqrt{(A - C)^2 + B^2}}{(A + C) + \sqrt{(A - C)^2 + B^2}} = \left( \sqrt{f} \gamma_s^2 p_v q_v \right)^{-2}, \quad (18)$$

which were derived by Stark (1977), though also see Binggeli (1980) and Binney (1985), where

$$A = \frac{\cos^2 \theta}{q_v^2} \left( \sin^2 \phi + \frac{\cos^2 \phi}{p_v^2} \right) + \frac{\sin^2 \theta}{p_v^2}, \quad (19)$$

$$B = \cos \theta \sin 2\phi \left( 1 - \frac{1}{p_v^2} \right) \frac{1}{q_v^2}, \quad (20)$$

$$C = \left( \frac{\sin^2 \phi}{p_v^2} + \cos^2 \phi \right) \frac{1}{q_v^2}. \quad (21)$$

<sup>1</sup> To convert our notation to that used by Stark (1977) for the projected quantities, let  $\gamma_s \rightarrow \alpha$ ,  $q_s \rightarrow \beta$ ,  $\alpha \rightarrow q$ , and  $X \leftrightarrow Y$ .

## 4.2 General Case

**Definition 1.** For an elliptical distribution  $\Sigma(\alpha)$ , where  $\alpha$  is the elliptical radius with axial ratio  $q_s$ , we define the “effective circular average”  $\langle \Sigma(R) \rangle$  by associating the circular radius with the geometric mean radius:  $\langle \Sigma(R) \rangle = \Sigma(\alpha)$ , where  $R = \alpha\sqrt{q_s}$ .

The use of such an effective circular average is required for obtaining the analytical relations we describe below. We note that for typical models (i.e., those considered in Paper 2) the effective circular average usually very closely approximates a formal azimuthal integration of  $\Sigma(\alpha)$  at fixed  $R$ . Only for very flattened models that fall steeply with radius do we find notable differences. We emphasize, however, that for our purposes it is only necessary that the observer employs the effective circular average in their analysis (as is a very common practice).

**Definition 2.** For any quantity intrinsically distributed as an ellipsoid stratified on surfaces of constant ellipsoidal radius  $a_v$ , let  $\langle \dots \rangle^d$  represent the deprojected, spherically averaged version of that quantity defined in the following manner. For illustration consider the specific case of  $\epsilon(a_v)$  and its projection  $\Sigma(a_s)$  as defined previously. The spherical averaging considered here begins by first taking the effective circular average of  $\Sigma(a_s)$  (i.e., Definition 1 and using the fact that  $a_s \propto \alpha$ ) to give the radial profile,  $\langle \Sigma(R) \rangle$ , where  $R$  is the radius on the sky. Then  $\langle \Sigma(R) \rangle$  is deprojected by assuming spherical symmetry, which yields the spherically averaged volume emissivity denoted by,  $\langle \epsilon(r) \rangle^d$ .

**Definition 3.** For any quantity that is composed of one or more deprojected spherically averaged quantities as defined in Definition 2, but itself does not satisfy Definition 2, we shall also refer to this composite quantity as a “deprojected spherical average” but add a plus sign in the notation to distinguish it from Definition 2; i.e.,  $\langle \dots \rangle^{d+}$ .

**Theorem 4.** For any ellipsoidal distribution that depends only on the ellipsoidal radius  $a_v$ , the deprojected spherical average (Definition 2) of this distribution is,

$$\langle \epsilon(r) \rangle^d = \left( \gamma_s q_s^{\frac{1}{2}} f^{-\frac{1}{2}} \right) \epsilon(a_v), \text{ where } a_v = (\gamma_s \sqrt{q_s}) r. \quad (22)$$

*Proof.* We start with  $\epsilon(a_v)$  and use eqn. (14) to compute  $\Sigma(a_s)$ . From Definition 1 the effective circular average of  $\Sigma(a_s)$  is obtained by associating the circular radius  $R$  with the geometric mean radius of the ellipse of semi-major axis  $\alpha$ ,  $R = \alpha\sqrt{q_s}$ . Since the ellipsoidal coordinate  $a_s$  is a function of the elliptical radius,  $a_s(\alpha) = \gamma_s q_s \alpha$ , it follows that,

$$\langle \Sigma(R) \rangle = \Sigma(a_s(\alpha)) = \Sigma\left(a_s\left(\frac{R}{\sqrt{q_s}}\right)\right) = \Sigma(\gamma_s \sqrt{q_s} R), \quad (23)$$

which relates the circular distribution on the L.H.S. to the elliptical distribution on the R.H.S. Since  $\langle \Sigma(R) \rangle$  is circularly symmetric, it may be spherically deprojected using the inverse Abel integral relation,

$$\langle \epsilon(r) \rangle^d = -\frac{1}{\pi} \int_r^\infty \frac{d\langle \Sigma(R) \rangle}{dR} \frac{dR}{\sqrt{R^2 - r^2}}. \quad (24)$$

Changing the integration variable from  $R$  to  $a_s$  yields,

$$\langle \epsilon(r) \rangle^d = -\frac{\gamma_s \sqrt{q_s}}{\pi} \int_{a_v}^\infty \frac{d\Sigma(a_s)}{da_s} \frac{da_s}{\sqrt{a_s^2 - a_v^2}}, \quad (25)$$

where  $a_v = (\gamma_s \sqrt{q_s}) r$ . Since eqn. (14) is an Abel integral, its inverse is readily obtained (e.g., eqn. 13 of Stark 1977),

$$\epsilon(a_v) = -\frac{\sqrt{f}}{\pi} \int_{a_v}^\infty \frac{d\Sigma(a_s)}{da_s} \frac{da_s}{\sqrt{a_s^2 - a_v^2}}. \quad (26)$$

Comparing equations (25) and (26) gives the desired result.  $\square$

## 4.3 X-Ray Emission and Hydrostatic Equilibrium

Here we consider cluster properties associated with the ICM X-ray emission. We assume that all volume ICM properties depend only on ellipsoidal radius  $a_v$ , which applies exactly for hydrostatic equilibrium in an EP (see §3.2). However, hydrostatic equilibrium is strictly required below only for Theorems 7 and 9 and Corollary 2.

**Theorem 5.** The deprojected spherical average of the emission-weighted temperature is,

$$\langle T(r) \rangle^{d+} = T(a_v), \text{ where } a_v = (\gamma_s \sqrt{q_s}) r. \quad (27)$$

*Proof.* Let  $\epsilon$  be the X-ray emissivity and  $T$  the gas temperature. The emission-weighted temperature is defined as the volume integral of  $\epsilon T$  divided by the volume integral of  $\epsilon$ . Hence, at any radius,

$$\langle T(r) \rangle^{d+} = \frac{\langle (\epsilon T)(r) \rangle^d}{\langle \epsilon(r) \rangle^d} = \frac{\left( \gamma_s q_s^{\frac{1}{2}} f^{-\frac{1}{2}} \right) \epsilon(a_v) T(a_v)}{\left( \gamma_s q_s^{\frac{1}{2}} f^{-\frac{1}{2}} \right) \epsilon(a_v)} = T(a_v), \quad (28)$$

where  $a_v = (\gamma_s \sqrt{q_s})r$ , and we made use of Theorem 4 in both the numerator and denominator. Note that the deprojected spherical averages in the numerator and denominator each correspond to Definition 2, while the composite result  $\langle T(r) \rangle^{d+}$  corresponds to Definition 3. Similar usages apply below.  $\square$

**Theorem 6.** The deprojected spherical average of the gas density is,

$$\langle \rho_{\text{gas}}(r) \rangle^{d+} = \left( \gamma_s^{\frac{1}{2}} q_s^{\frac{1}{4}} f^{-\frac{1}{4}} \right) \rho_{\text{gas}}(a_v), \text{ where } a_v = (\gamma_s \sqrt{q_s})r. \quad (29)$$

*Proof.* From the definition of the X-ray emissivity,

$$\langle \rho_{\text{gas}}(r) \rangle^{d+} = \left[ \frac{\langle \epsilon(r) \rangle^d}{\Lambda(\langle T(r) \rangle^{d+})} \right]^{\frac{1}{2}}, \quad (30)$$

where we have suppressed the metallicity dependence of the plasma emissivity  $\Lambda$ , which does not affect our arguments provided that the metallicity depends only on  $a_v$  as we are assuming for all ICM properties. Applying Theorem 4 in the numerator and Theorem 5 in the denominator of the above equation yields,

$$\langle \rho_{\text{gas}}(r) \rangle^{d+} = \left[ \frac{\left( \gamma_s q_s^{\frac{1}{2}} f^{-\frac{1}{2}} \right) \epsilon(a_v)}{\Lambda(T(a_v))} \right]^{\frac{1}{2}} = \left[ \frac{\left( \gamma_s q_s^{\frac{1}{2}} f^{-\frac{1}{2}} \right) \rho_{\text{gas}}^2(a_v) \Lambda(T(a_v))}{\Lambda(T(a_v))} \right]^{\frac{1}{2}}, \quad (31)$$

where  $a_v = (\gamma_s \sqrt{q_s})r$ , which reduces to the desired result.  $\square$

**Corollary 1.** The deprojected spherical averages of the gas pressure and entropy are,

$$\langle P_{\text{gas}}(r) \rangle^{d+} = \left( \gamma_s^{\frac{1}{2}} q_s^{\frac{1}{4}} f^{-\frac{1}{4}} \right) P_{\text{gas}}(a_v), \quad \langle S(r) \rangle^{d+} = \left( \gamma_s^{-\frac{1}{3}} q_s^{-\frac{1}{6}} f^{\frac{1}{6}} \right) S(a_v), \text{ where } a_v = (\gamma_s \sqrt{q_s})r. \quad (32)$$

*Proof.* These results are immediate consequences of the definitions of each quantity,  $P_{\text{gas}} = \rho_{\text{gas}} k_B T / (\mu m_a)$  and  $S = (k_B / \mu m_a) T \rho_{\text{gas}}^{-2/3}$ , Theorem 5, and Theorem 6.  $\square$

**Theorem 7.** The deprojected spherical average of the total mass enclosed within radius  $r$  is,

$$\langle M(< r) \rangle^{d+} = \left( \gamma_s q_s^{\frac{1}{2}} \eta(p_v, q_v) \right)^{-1} M(< a_v), \text{ where } a_v = (\gamma_s \sqrt{q_s})r, \quad (33)$$

for any temperature profile.

*Proof.* Applying hydrostatic equilibrium for a spherically symmetric cluster gives,

$$\langle M(< r) \rangle^{d+} = \frac{-1}{\langle \rho_{\text{gas}}(r) \rangle^{d+}} \frac{r^2}{G} \frac{d}{dr} \langle P_{\text{gas}}(r) \rangle^{d+} = \frac{-1}{\rho_{\text{gas}}(a_v)} \frac{r^2}{G} \frac{d}{dr} P_{\text{gas}}(a_v), \quad (34)$$

where  $a_v = (\gamma_s \sqrt{q_s})r$ , and the R.H.S. made use of Corollary 1, Theorem 6, and the fact that  $\gamma_s$ ,  $q_s$ , and  $f$  depend only on  $p_v$ ,  $q_v$ , and the fixed line-of-sight projection orientation. Changing the variable from  $r$  to  $a_v = (\gamma_s \sqrt{q_s})r$  so that  $d/dr = (\gamma_s \sqrt{q_s})d/da_v$ , after simplifying, gives,

$$\langle M(< r) \rangle^{d+} = \frac{1}{\gamma_s \sqrt{q_s}} \left[ \frac{-1}{\rho_{\text{gas}}(a_v)} \frac{a_v^2}{G} \frac{d}{da_v} P_{\text{gas}}(a_v) \right]. \quad (35)$$

By making use of eqn. (12) and Theorem 1 the quantity in brackets equals  $M(< a_v)/\eta(p_v, q_v)$ , which proves the theorem, for any temperature profile  $T(a_v)$ .  $\square$

We have chosen to state explicitly that this result for the total mass holds for any temperature profile since we desire to emphasize this point below in Theorem 9. Next, however, we consider the gas mass.

**Theorem 8.** The deprojected spherical average of the gas mass enclosed within radius  $r$  is,

$$\langle M_{\text{gas}}(< r) \rangle^{d+} = \left( \gamma_s^{\frac{5}{2}} q_s^{\frac{5}{4}} f^{\frac{1}{4}} p_v q_v \right)^{-1} M_{\text{gas}}(< a_v), \text{ where } a_v = (\gamma_s \sqrt{q_s})r. \quad (36)$$

*Proof.* The gas mass within a spherical volume of radius  $r$  is,

$$\langle M_{\text{gas}}(< r) \rangle^{d+} = \int_0^r \langle \rho_{\text{gas}}(r) \rangle^{d+} 4\pi r^2 dr = \left( \gamma_s^{\frac{1}{2}} q_s^{\frac{1}{4}} f^{-\frac{1}{4}} \right) \int_0^r \rho_{\text{gas}}(a_v) 4\pi r^2 dr, \quad (37)$$

where  $a_v = (\gamma_s \sqrt{q_s})r$ , and in the R.H.S. we applied Theorem 6 and again (as in Theorem 7) made use of the fact that  $\gamma_s$ ,  $q_s$ , and  $f$  do not depend on  $r$ . Changing the integration variable from  $r$  to  $a_v = (\gamma_s \sqrt{q_s})r$ , and simplifying, gives,

$$\langle M_{\text{gas}}(< r) \rangle^{d+} = \left( \gamma_s^{\frac{5}{2}} q_s^{\frac{5}{4}} f^{\frac{1}{4}} \right)^{-1} \int_0^{a_v} \rho_{\text{gas}}(a_v) 4\pi a_v^2 da_v. \quad (38)$$

Since an ellipsoidal volume element is  $dV = 4\pi p_v q_v a_v^2 da_v$ , the integral equals  $M_{\text{gas}}(< a_v)/(p_v q_v)$ , which proves the theorem.  $\square$

**Corollary 2.** The deprojected spherical average of the gas mass fraction enclosed within radius  $r$  is,

$$\langle f_{\text{gas}}(< r) \rangle^{\text{d}+} = \left( \gamma_s^{\frac{3}{2}} q_s^{\frac{3}{4}} f^{\frac{1}{4}} p_v q_v / \eta(p_v, q_v) \right)^{-1} f_{\text{gas}}(< a_v), \text{ where } a_v = (\gamma_s \sqrt{q_s}) r. \quad (39)$$

*Proof.* This result follows immediately from the definition of the gas fraction,  $f_{\text{gas}}(< r) = M_{\text{gas}}(< r)/M(< r)$ , Theorem 7, and Theorem 8.  $\square$

**Theorem 9.** For the scale-free logarithmic EP,  $\Phi(a_v) = (GM_{\Delta}/a_{\Delta}) \ln(a_v)$ , spherical averaging does not bias the mass profile in the sense that,

$$\langle M(< r) \rangle^{\text{d}+} = \langle M(< r) \rangle^{\text{true}}, \quad (40)$$

where  $\langle M(< r) \rangle^{\text{true}}$  is the spherical average of true mass distribution corresponding to  $\Phi(a_v)$ . This result is independent of the gas temperature profile.

*Proof.* This is a special case of the general problem that is the focus of Paper 2; i.e., we wish to compare spherically averaged quantities obtained by an observer to those obtained by theoretical studies. While an observer will measure deprojected spherically averaged quantities (Definitions 2 and 3), the theorist typically spherically averages the true three-dimensional distribution directly. From Theorem 2, we obtain the spherical average of the mass distribution generated by the scale-free logarithmic EP,  $\Phi(a_v) = (GM_{\Delta}/a_{\Delta}) \ln(a_v)$ , where  $M_{\Delta}$  and  $a_{\Delta}$  are constants,

$$\langle M(< r) \rangle^{\text{true}} = \frac{r}{4\pi G} \int_{4\pi} a_v \frac{d\Phi(a_v)}{da_v} d\Omega = M_{\Delta} \frac{r}{a_{\Delta}}. \quad (41)$$

For comparison, the deprojected spherical average of the mass profile (Theorem 7) depends on the mass enclosed within ellipsoidal radius  $a_v$ , which is obtained by inserting the definition of the scale-free  $\Phi(a_v)$  into eqn. (7) of Theorem 1,

$$M(< a_v) = \eta(p_v, q_v) M_{\Delta} \frac{a_v}{a_{\Delta}} = \left( \gamma_s q_s^{\frac{1}{2}} \eta(p_v, q_v) \right) M_{\Delta} \frac{r}{a_{\Delta}}, \quad (42)$$

where the R.H.S. made use of the substitution  $a_v = (\gamma_s \sqrt{q_s}) r$  as appropriate for Theorem 7. Now substituting  $M(< a_v)$  into Theorem 7 gives,  $\langle M(< r) \rangle^{\text{d}+} = M_{\Delta} r / a_{\Delta} = \langle M(< r) \rangle^{\text{true}}$ , independent of the gas temperature profile, which proves the theorem.  $\square$

This theorem complements and extends the result presented in Appendix B of Churazov et al. (2008). These authors consider the bias due to spherical averaging of an isothermal ICM with a scale-free gas density,  $\rho_{\text{gas}} = h(\theta, \phi) r^{-\alpha}$ , where  $h(\theta, \phi)$  is some positive function. They argue that this gas density distribution leads to an inferred total mass that also displays no bias due to spherical averaging. Because their model implies a potential,  $\Phi \propto \ln[h(\theta, \phi) r^{-\alpha}]$ , equivalent to the scale-free potential we employed above in Theorem 9 for  $\alpha = 1$  and  $h = 1/\sqrt{f}$ , it is reassuring that the two results each predict no bias for the special case of an isothermal ICM. The assumption of the scale-free EP with no *a priori* restriction on the form of the gas density has allowed us to generalize rigorously the zero-bias result for any ICM temperature profile.

#### 4.4 Sunyaev-Zel'dovich Effect and Related X-ray Quantities

Now we consider a galaxy cluster also to be observed via the thermal SZ effect, and we continue to assume that all three-dimensional ICM properties depend only on the ellipsoidal radius  $a_v$ .

**Corollary 3.** The deprojected spherical average of the ICM electron pressure,  $P_e = n_e k_B T$ , obtained from a measurement of the thermal SZ effect is,

$$\langle P_e(r) \rangle_{\text{SZ}}^{\text{d}} = \left( \gamma_s q_s^{\frac{1}{2}} f^{-\frac{1}{2}} \right) P_e(a_v), \text{ where } a_v = (\gamma_s \sqrt{q_s}) r. \quad (43)$$

*Proof.* The thermal SZ effect is the temperature decrement,  $\delta T/T = -2y_c$  in the Rayleigh-Jeans tail of the Cosmic Background Radiation (CBR) spectrum due to inverse Compton scattering of CBR photons by energetic ICM electrons. The Compton-y parameter is,

$$y_c = \frac{\sigma_T}{m_e c^2} \int_{\text{los}} P_e dz' = \frac{\sigma_T}{m_e c^2} \frac{2}{\sqrt{f}} \int_{a_s}^{\infty} \frac{P_e(a_v) a_v da_v}{\sqrt{a_v^2 - a_s^2}}, \quad (44)$$

where  $\sigma_T$  is the Thomson cross section,  $m_e$  is the electron mass, and the R.H.S. follows from the condition that  $P_e = P_e(a_v)$  (i.e., eqn. 14). By associating  $y_c(a_s)$  with  $\Sigma(a_s)$  and  $\sigma_T P_e(a_v)/(m_e c^2)$  with  $\epsilon(a_v)$  the result follows immediately from Theorem 4.  $\square$



The geometrical factor  $\gamma_s q_s^{\frac{1}{2}} f^{-\frac{1}{2}}$  in eqn. (43) is the square of the corresponding factor for the ICM pressure obtained from X-ray studies (Corollary 1), indicating that spherical averaging has a stronger impact on the ICM pressure inferred from SZ studies. Since this difference in geometrical factors can be exploited to measure the intrinsic shape and orientation of a cluster, we state it formally.

**Corollary 4.** The ratio of the deprojected spherical averages of the ICM electron pressures inferred from SZ and X-ray studies is,

$$\langle P_{\text{SZ,X}}(r) \rangle^{\text{d}+} \equiv \frac{\langle P_e(r) \rangle_{\text{SZ}}^{\text{d}}}{\langle P_e(r) \rangle_{\text{X}}^{\text{d}+}} = \gamma_s^{\frac{1}{2}} q_s^{\frac{1}{4}} f^{-\frac{1}{4}}. \quad (45)$$

*Proof.* This result follows immediately from Corollaries 1 and 3, where we have used the ICM electron pressure from X-rays,  $\langle P_e(r) \rangle_{\text{X}}^{\text{d}+} = ((2 + \mu)/5) \langle P_{\text{gas}}(r) \rangle^{\text{d}+}$ .  $\square$

We remark that  $\langle P_{\text{SZ,X}}(r) \rangle^{\text{d}+}$  is, in fact, constant with radius and does not overtly display any dependence on the distance to the cluster, and hence the Hubble Constant,  $H_0$ . However,  $P_{\text{X}} \propto \rho_{\text{gas}} \propto \sqrt{\epsilon}$ , where  $\epsilon$  is the volume emissivity. To obtain physical units for the emissivity requires converting the observed X-ray flux to a luminosity density, the net result of which is that  $P_{\text{X}}$  is inversely proportional to the square root of the cluster distance, leading to  $\langle P_{\text{SZ,X}}(r) \rangle^{\text{d}+} \propto 1/\sqrt{H_0}$ .

The possibility of uncovering the intrinsic shape of the cluster ICM by combining X-ray and SZ measurements has been recognized for over ten years (e.g., Zaroubi et al. 1998; Fox & Pen 2002; Reblinsky 2000; Lee & Suto 2004; De Filippis et al. 2005; Puchwein & Bartelmann 2006; Sereno 2007; Mahdavi & Chang 2011). This promising technique has already provided interesting constraints on cluster shapes for many clusters (De Filippis et al. 2005; Sereno et al. 2006) using isothermal triaxial  $\beta$  models for the ICM, and more recently models with a radially varying temperature profile have been applied to the cluster A 1689 (Sereno et al. 2011). Corollary 4 defines a particular approach to this problem that has some attractive characteristics. First, the ratio of spherically averaged pressures is valid for any temperature profile  $T(a_v)$ . Second, the relationship does not assume a particular ICM radial density profile (e.g.,  $\beta$  model) and, in principle, can be deprojected using the traditional spherical onion peeling procedure. Hence, studies can be conducted entirely in the context of spherical symmetry to obtain  $\langle P_{\text{SZ,X}}(r) \rangle^{\text{d}+}$  and then, supplemented with a measurement of the average ICM axial ratio on the sky ( $q_s$ ), can be used to constrain the geometrical factor  $\gamma_s^{\frac{1}{2}} q_s^{\frac{1}{4}} f^{-\frac{1}{4}}$ . For the general triaxial ellipsoid this factor depends on the intrinsic shape via the axial ratios  $p_v$  and  $q_v$  and the orientation  $(\theta, \phi)$ . For spheroids only a single axial ratio and inclination angle are required.

We now consider the quantity,

$$Y_{\text{SZ}} \equiv \frac{1}{D_A^2(z)} \frac{\sigma_T}{m_e c^2} \int_V P_e dV, \quad (46)$$

where  $V$  is the volume and  $D_A(z)$  is the angular diameter distance to the cluster. Since this quantity equals the integral of  $y_c$  over solid angle in the limit of a small angle subtended on the sky, it is usually referred to as the “integrated Compton-y parameter” (e.g., White et al. 2002). We will take  $V$  to be a large spherical or ellipsoidal region centered on the cluster.

**Theorem 10.** The deprojected spherical average of the integrated Compton-y parameter is,

$$\langle Y_{\text{SZ}}(< r) \rangle^{\text{d}+} = \left( \gamma_s^2 q_s f^{\frac{1}{2}} p_v q_v \right)^{-1} Y_{\text{SZ}}(< a_v), \text{ where } a_v = (\gamma_s \sqrt{q_s}) r. \quad (47)$$

*Proof.* Starting from the definition of  $Y_{\text{SZ}}$ , we have for a spherical volume,

$$\langle Y_{\text{SZ}}(< r) \rangle^{\text{d}+} = \frac{1}{D_A^2(z)} \frac{\sigma_T}{m_e c^2} \int_0^r \langle P_e(r) \rangle_{\text{SZ}}^{\text{d}} 4\pi r^2 dr \quad (48)$$

$$= \left( \gamma_s q_s^{\frac{1}{2}} f^{-\frac{1}{2}} \right) \frac{1}{D_A^2(z)} \frac{\sigma_T}{m_e c^2} \int_0^r P_e(a_v) 4\pi r^2 dr, \quad (49)$$

where we have used Corollary 3 and the fact that  $\gamma_s$ ,  $q_s$ , and  $f$  depend only on  $p_v$ ,  $q_v$ , and the fixed line-of-sight projection orientation. Effecting a change of variable within the integral from  $r$  to  $a_v = (\gamma_s \sqrt{q_s}) r$  yields,

$$\langle Y_{\text{SZ}}(< r) \rangle^{\text{d}+} = \left( \gamma_s^2 q_s f^{\frac{1}{2}} \right)^{-1} \frac{1}{D_A^2(z)} \frac{\sigma_T}{m_e c^2} \int_0^{a_v} P_e(a_v) 4\pi a_v^2 da_v \quad (50)$$

$$= \left( \gamma_s^2 q_s f^{\frac{1}{2}} \right)^{-1} \left[ \frac{Y_{\text{SZ}}(< a_v)}{p_v q_v} \right], \quad (51)$$

where the last step used the definition of an ellipsoidal volume element,  $dV = 4\pi p_v q_v a_v^2 da_v$ , which proves the theorem.  $\square$

Cosmological simulations predict a strong correlation between  $Y_{\text{SZ}}$  and cluster mass (e.g., White et al. 2002), which is a direct result of the gas pressure probing the depth of the cluster potential well. However, because simulations do not perfectly match observations of cluster ICM (e.g., isophotal flattening of cool cores, Fang et al. 2009), and since it is preferable to

use an independent method to constrain cosmology, there is interest in using X-ray observations of cluster mass to calibrate  $Y_{\text{SZ}}$  independently. For high-quality X-ray data of clusters where it is possible to measure accurately the spatially resolved gas density and temperature profiles, direct calculation of the ICM pressure profile is to be preferred for comparison to  $Y_{\text{SZ}}$  (Arnaud et al. 2010). For lower quality data it is necessary to rely on scaling relations, such as  $Y_X$ , a quantity advocated by Kravtsov et al. (2006) as a mass proxy for cosmological studies,

$$Y_X(< r) = M_{\text{gas}}(< r) T_X(< r), \quad (52)$$

where  $M_{\text{gas}}(< r)$  is the gas mass and  $T_X(< r)$  is the emission-weighted temperature enclosed within the spherical volume of radius  $r$ , and typically a radius  $r_{500}$  is adopted. Since  $Y_X$  is related to the integrated gas pressure profile, it should be closely related to  $Y_{\text{SZ}}$ . Indeed, recent SZ studies find a strong correlation between  $Y_{\text{SZ}}(< r_{500})$  and  $Y_X(< r_{500})$  and between  $Y_{\text{SZ}}(< r_{500})$  and  $M_{\text{gas}}(< r_{500})$ , each correlation having similar intrinsic scatter (e.g., Andersson et al. 2010; Planck Collaboration et al. 2011) – see also Fabjan et al. (2011).

Before addressing the spherical average of  $Y_X$ , we consider  $T_X$ .

**Theorem 11.** The deprojected spherical average of the emission-weighted temperature integrated over the spherical volume of radius  $r$  is,

$$\langle T_X(< r) \rangle^{\text{d}+} = T_X(< a_v), \text{ where } a_v = (\gamma_s \sqrt{q_s}) r. \quad (53)$$

*Proof.* Beginning with the definition of  $\langle T_X(< r) \rangle^{\text{d}+}$  as the deprojected integrated emission-weighted temperature, we have,

$$\langle T_X(< r) \rangle^{\text{d}+} = \frac{\int_0^r \langle (\epsilon T)(r) \rangle^{\text{d}} 4\pi r^2 dr}{\int_0^r \langle \epsilon(r) \rangle^{\text{d}} 4\pi r^2 dr} = \frac{\int_0^r \epsilon(a_v) T(a_v) 4\pi r^2 dr}{\int_0^r \epsilon(a_v) 4\pi r^2 dr}, \quad (54)$$

where we made use of Theorem 4 in both the numerator and denominator and the fact that the factor  $(\gamma_s \sqrt{q_s}/f)$  depends only on  $p_v$ ,  $q_v$ , and the fixed line-of-sight projection orientation. Effecting a change of variable within the integrals from  $r$  to  $a_v = (\gamma_s \sqrt{q_s}) r$  yields,

$$\langle T_X(< r) \rangle^{\text{d}+} = \frac{\int_0^{a_v} \epsilon(a_v) T(a_v) 4\pi a_v^2 da_v}{\int_0^{a_v} \epsilon(a_v) 4\pi a_v^2 da_v}. \quad (55)$$

Multiplying the numerator and denominator of the R.H.S. by  $p_v q_v$  yields the emission-weighted temperature integrated over the volume of ellipsoidal radius  $a_v = (\gamma_s \sqrt{q_s}) r$ , which is the desired result.  $\square$

We remark that Theorem 11 reduces to Theorem 5 for the special case of a small radial volume element associated with a finite radius  $r$ .

**Corollary 5.** The deprojected spherical average of  $Y_X(< r)$  is,

$$\langle Y_X(< r) \rangle^{\text{d}+} = \left( \gamma_s^{\frac{5}{2}} q_s^{\frac{5}{4}} f^{\frac{1}{4}} p_v q_v \right)^{-1} Y_X(< a_v), \text{ where } a_v = (\gamma_s \sqrt{q_s}) r. \quad (56)$$

*Proof.* This result is an immediate consequence of the definition of  $Y_X(< r)$  (eqn. 52) and Theorems 8 and 11.  $\square$

**Corollary 6.** The ratio of the deprojected spherical averages of  $Y_{\text{SZ}}$  and  $Y_X$  is,

$$\langle Y_{\text{SZ},X}(< r) \rangle^{\text{d}+} \equiv \frac{\langle Y_{\text{SZ}}(< r) \rangle^{\text{d}+}}{\langle Y_X(< r) \rangle^{\text{d}+}} = \left( \gamma_s^{\frac{1}{2}} q_s^{\frac{1}{4}} f^{-\frac{1}{4}} \right) \frac{Y_{\text{SZ}}(< a_v)}{Y_X(< a_v)} = \left( \gamma_s^{\frac{1}{2}} q_s^{\frac{1}{4}} f^{-\frac{1}{4}} \right) Y_{\text{SZ},X}(< a_v), \text{ where } a_v = (\gamma_s \sqrt{q_s}) r. \quad (57)$$

*Proof.* This result follows immediately from Theorem 10 and Corollary 5.  $\square$

#### 4.5 Connection to Stellar Dynamics

Many of the results we have presented can be applied either directly, or with minor modification, to a relaxed, dispersion-supported collisionless stellar system with an isotropic velocity dispersion tensor. Such a system obeys the equation of hydrostatic equilibrium where the stellar density  $\rho_{\text{stars}}(a_v)$  replaces the ICM density and the square of the velocity dispersion  $\sigma(a_v)^2$  replaces the gas temperature. (Here it is assumed the stars, like the gas, are merely a tracer of the gravitational potential, which is a good approximation for galaxy clusters, and also for elliptical galaxies well outside of the stellar half-light radius.) Consequently,  $\langle \rho_{\text{stars}}(r) \rangle^{\text{d}}$  obeys Theorem 4, and  $\langle \sigma(< r)^2 \rangle^{\text{d}+}$  obeys Theorem 5. Similarly, the deprojected spherically averaged mass inferred from the stellar dynamics also obeys Theorem 7, where in the proof one replaces the gas density with  $\rho_{\text{stars}}(a_v)$  and the pressure with  $\rho_{\text{stars}}(a_v) \sigma(a_v)^2$ . Finally, the deprojected spherically averaged stellar mass profile behaves as  $\langle Y_{\text{SZ}}(< r) \rangle^{\text{d}+}$  (Theorem 10), because the deprojected stellar mass density behaves as  $\langle P_e(r) \rangle_{\text{SZ}}^{\text{d}}$  (Corollary 3).

## 5 PROJECTION OF ELLIPSOIDAL SHELLS AND ONION PEELING DEPROJECTION

To treat the case of binned observational data, such as the one-dimensional surface brightness profile of a cluster, here we describe the projection and deprojection of a system of concentric, similar triaxial ellipsoidal shells relevant for the study of EPs. For an ellipsoidal shell defined between  $a_v^{\text{in}}$  and  $a_v^{\text{out}}$  with constant emissivity,  $\epsilon(a_v^{\text{in}}, a_v^{\text{out}})$ , throughout the shell, equation (14) becomes,

$$\Sigma(a_v^{\text{in}}, a_v^{\text{out}}; a_s) = \frac{2\epsilon(a_v^{\text{in}}, a_v^{\text{out}})}{\sqrt{f}} \left[ \sqrt{(a_v^{\text{out}})^2 - a_s^2} - \sqrt{(a_v^{\text{in}})^2 - a_s^2} \right], \quad (58)$$

where  $a_s \leq a_v^{\text{out}}$ , and the second term in brackets is set to zero if  $a_s > a_v^{\text{in}}$ . This equation projects a three-dimensional ellipsoidal shell onto a two-dimensional elliptical surface brightness that depends only on the elliptical coordinate  $a_s$ . We desire the luminosity integrated over an elliptical annulus defined between semi-major axes,  $\alpha^{\text{in}}$  and  $\alpha^{\text{out}}$ :

$$L(a_v^{\text{in}}, a_v^{\text{out}}; \alpha^{\text{in}}, \alpha^{\text{out}}) = \int_{\alpha^{\text{in}}}^{\alpha^{\text{out}}} \Sigma(a_v^{\text{in}}, a_v^{\text{out}}; a_s) 2\pi q_s \alpha d\alpha \quad (59)$$

$$= \epsilon(a_v^{\text{in}}, a_v^{\text{out}}) V^{\text{int}}(a_v^{\text{in}}, a_v^{\text{out}}; \alpha^{\text{in}}, \alpha^{\text{out}}), \quad (60)$$

where,

$$V^{\text{int}}(a_v^{\text{in}}, a_v^{\text{out}}; \alpha^{\text{in}}, \alpha^{\text{out}}) = \frac{4\pi}{3} p_v q_v \times \left( \left[ (a_v^{\text{out}})^2 - (\gamma_s q_s \alpha^{\text{in}})^2 \right]^{3/2} - \left[ (a_v^{\text{out}})^2 - (\gamma_s q_s \alpha^{\text{out}})^2 \right]^{3/2} + \left[ (a_v^{\text{in}})^2 - (\gamma_s q_s \alpha^{\text{out}})^2 \right]^{3/2} - \left[ (a_v^{\text{in}})^2 - (\gamma_s q_s \alpha^{\text{in}})^2 \right]^{3/2} \right). \quad (61)$$

If any terms in equation (61) have negative arguments, they must be set to zero. Note when viewed “edge-on” ( $\theta = \phi = 90^\circ$ ), i.e., down the intermediate principal axis, ellipsoids always have  $q_s = q_v$  and  $\gamma_s q_s = 1$ . For the special case of an oblate spheroid ( $p_v = 1$ ) viewed at arbitrary inclination,  $q_s = q_v \sqrt{f}$  and  $\gamma_s q_s = 1$ .

Let an ellipsoid be partitioned into a series of concentric, similar ellipsoidal shells,  $a_{v,0} < a_{v,1} < a_{v,2} < \dots < a_{v,N}$ . Define a corresponding set of concentric, similar, elliptical annuli such that,  $\alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_N$ , where  $\alpha_0 = a_{v,0}$ ,  $\alpha_1 = a_{v,2}, \dots$ . For this case, we may represent the projection of the three-dimensional ellipsoidal shell ( $a_{j-1}, a_j$ ) onto the two-dimensional elliptical annulus ( $\alpha_{i-1}, \alpha_i$ ) by,

$$V_{ji}^{\text{int}} \equiv V^{\text{int}}(a_{v,j-1}, a_{v,j}; \alpha_{i-1}, \alpha_i). \quad (62)$$

That is, each shell and annulus is labeled by the index of its outer boundary. The contribution of shell  $j$  to the luminosity of annulus  $i$  is,  $L_i = \epsilon_j V_{ji}^{\text{int}}$ , where  $\epsilon_j \equiv \epsilon(a_{v,j-1}, a_{v,j})$  is the constant emissivity within the shell. We obtain the total luminosity projected into annulus  $i$  by summing the contributions from all shells  $j \geq i$ ,

$$L_i = \sum_{j=i}^N \epsilon_j V_{ji}^{\text{int}}, \quad (63)$$

so that the surface brightness is,

$$\Sigma_i = \frac{L_i}{\pi q_s (\alpha_i^2 - \alpha_{i-1}^2)} = \frac{1}{\pi q_s (\alpha_i^2 - \alpha_{i-1}^2)} \sum_{j=i}^N \epsilon_j V_{ji}^{\text{int}}. \quad (64)$$

Hence, we have shown that the projection matrix for spherical shells (e.g., equation B12 of Gastaldello et al. 2007) is generalized to the case of ellipsoidal symmetry via the following mapping: three-dimensional radius,  $r \rightarrow a_v$ ; two-dimensional radius,  $R \rightarrow a_s = \gamma_s q_s \alpha$ ; and  $4\pi/3 \rightarrow p_v q_v 4\pi/3$ . Moreover, by separating the first term from the others in the summation of eqn. (63),

$$L_i = \epsilon_i V_{ii}^{\text{int}} + \sum_{j=i+1}^N \epsilon_j V_{ji}^{\text{int}}, \quad (65)$$

and then solving for the emissivity in shell  $i$ ,

$$\epsilon_i = \left( \frac{L_i}{V_{ii}^{\text{int}}} \right) - \sum_{j=i+1}^N \epsilon_j \left( \frac{V_{ji}^{\text{int}}}{V_{ii}^{\text{int}}} \right), \quad (66)$$

we arrive at a generalization of the “onion peeling” deprojection method (Fabian et al. 1981; Kriss et al. 1983) appropriate for triaxial ellipsoids ( $p_v, q_v$ ) with any orientation ( $\theta, \phi$ ). That is, the emissivity in shell  $i$  is obtained by taking the total luminosity observed in annulus  $i$  on the sky, subtracting from it the luminosity contributions projected from shells at larger radii ( $j > i$ ), and then dividing by  $V_{ii}^{\text{int}}$  representing the volume of intersection between the ellipsoidal shell  $i$  and the ellipsoidal cylindrical

shell defined by the base of elliptical annulus  $i$  and an infinite height. Practical implementation of eqn. (66) requires assuming values for  $\theta$ ,  $\phi$ ,  $p_v$ , and  $q_v$  and measuring the value of  $q_s$  on the sky.

While the above has focused on the example of the emissivity projecting into the surface brightness, generalization to other quantities such as the emission-weighted temperature and projected temperature map ( $\langle T \rangle_i$ ) is straightforward; i.e.,

$$\langle T \rangle_i = \frac{1}{L_i} \sum_{j=i}^N \epsilon_j T_j V_{ji}^{\text{int}}. \quad (67)$$

## 6 CONCLUSIONS

This is the first of two papers investigating the deprojection and spherical averaging of ellipsoidal galaxy clusters (and massive elliptical galaxies). We specifically consider applications to X-ray and SZ studies, though many of the results also apply to isotropic dispersion-supported stellar dynamical systems. A major disadvantage of working with ellipsoidal systems, as opposed to spherical systems, is that they generally involve numerical evaluation of computationally expensive integrals. Here we present analytical formulas for galaxy clusters described by a gravitational potential that is a triaxial ellipsoid of constant shape and orientation; i.e., an “ellipsoidal potential” (EP), which depends only on ellipsoidal radius,  $\Phi = \Phi(a_v)$ . While the mass density is itself not ellipsoidal for these models, and it can take unphysical values in the vicinity of the minor axis when the flattening is too large, we demonstrate that the total mass enclosed within the ellipsoidal radius  $a_v$  is proportional to  $d\Phi/da_v$ , and is therefore well-behaved for any smooth  $\Phi(a_v)$ , making it useful for many purposes.

We show that for hydrostatic X-ray studies of EPs the relationship between the enclosed total mass, ICM temperature, and ICM density has the same form (up to a proportionality factor) as the spherical case where the spherical radius  $r$  is replaced by  $a_v$ . Using this result, along with the general result we derive for the spherical deprojection of any ellipsoid of constant shape and orientation, we show that the mass bias due to spherically averaging X-ray observations is independent of the temperature profile. For the special case of a scale-free logarithmic EP ( $\Phi \propto \ln a_v$ ) there is exactly zero bias for any shape, orientation, and temperature profile. The ratio of spherically averaged ICM pressures obtained from SZ and X-ray measurements depends only on the intrinsic shape and projection orientation of the EP, as well as  $H_0$ , which provides another illustration of how cluster geometry can be recovered through a combination of X-ray and SZ measurements, with the key advantage that the pressures are measured in the context of spherical symmetry without (in principle) having to specify a parametric form for the radial profile. We also demonstrate that  $Y_{\text{SZ}}$  and  $Y_{\text{X}}$  have different biases as a result of spherical averaging, which lead to an offset in the spherically averaged  $Y_{\text{SZ}} - Y_{\text{X}}$  relation. Paper 2 explores in more detail the biases and scatter arising from spherical averaging, in particular using the observationally and cosmologically motivated NFW mass profile, and also considers the more widely investigated, and computationally expensive, class of potentials where the mass density, rather than the potential itself, is an ellipsoid of constant shape and orientation.

A potentially useful application of these analytical formulas is to assess the error range on an observable accounting for deviations from assumed spherical symmetry without having to perform the ellipsoidal deprojection explicitly. That is, an X-ray observer can, as is standard, analyze a cluster assuming spherical symmetry and obtain deprojected ICM temperature and density profiles using the spherical onion peeling procedure (see §5; Fabian et al. 1981; Kriss et al. 1983) to construct, e.g., the observed spherically averaged mass profile,  $\langle M(< r) \rangle^{\text{d+}}$ . With the aid of Theorem 7 this can be converted into the true mass profile  $M(< a_v)$  for an assumed three-dimensional shape and viewing orientation. Then by using Theorem 2 the true spherically averaged mass profile  $\langle M(< r) \rangle^{\text{true}}$  can be constructed. By adopting priors for the shape and orientation and marginalizing over them, the full range of  $\langle M(< r) \rangle^{\text{true}}$  owing to intrinsic ellipsoidal geometry can be computed.

Finally, for dedicated ellipsoidal studies, we also generalize the spherical onion peeling method to the triaxial case for a given shape and orientation. The formulas presented for ellipsoidal shells may also be of use for ellipsoidal projections in numerical work.

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## REFERENCES

Allen S. W., Evrard A. E., Mantz A. B., 2011, ArXiv e-prints

- Allgood B., Flores R. A., Primack J. R., Kravtsov A. V., Wechsler R. H., Faltenbacher A., Bullock J. S., 2006, MNRAS, 367, 1781
- Andersson K., Benson B. A., Ade P. A. R., Aird K. A., Armstrong B., Bautz M., Bleem L. E., Brodwin M., Carlstrom J. E., Chang C. L., et al. 2010, ArXiv e-prints
- Arnaud M., Pratt G. W., Piffaretti R., Böhringer H., Croston J. H., Pointecouteau E., 2010, A&A, 517, A92+
- Bailin J., Steinmetz M., 2005, ApJ, 627, 647
- Bett P., Eke V., Frenk C. S., Jenkins A., Helly J., Navarro J., 2007, MNRAS, 376, 215
- Binggeli B., 1980, A&A, 82, 289
- Binney J., 1981, MNRAS, 196, 455
- Binney J., 1985, MNRAS, 212, 767
- Binney J., Tremaine S., 2008, Galactic Dynamics: Second Edition. Princeton University Press
- Blandford R. D., Kochanek C. S., 1987, ApJ, 321, 658
- Bullock J. S., 2002, in P. Natarajan ed., The Shapes of Galaxies and their Dark Halos Shapes of Dark Matter Halos. pp 109–113
- Buote D. A., Canizares C. R., 1992, ApJ, 400, 385
- Buote D. A., Canizares C. R., 1994, ApJ, 427, 86
- Buote D. A., Canizares C. R., 1996a, ApJ, 457, 177
- Buote D. A., Canizares C. R., 1996b, ApJ, 457, 565
- Buote D. A., Humphrey P. J., 2011a, in D.-W. Kim, & S. Pellegrini ed., Hot Interstellar Matter in Elliptical Galaxies Springer (arXiv1104.0012), Dark Matter in Elliptical Galaxies
- Buote D. A., Humphrey P. J., 2011b, MNRAS, submitted (arXiv:1109.6656)
- Buote D. A., Jeltrema T. E., Canizares C. R., Garmire G. P., 2002, ApJ, 577, 183
- Buote D. A., Tsai J. C., 1995, ApJ, 439, 29
- Chandrasekhar S., 1987, Ellipsoidal figures of equilibrium
- Churazov E., Forman W., Vikhlinin A., Tremaine S., Gerhard O., Jones C., 2008, MNRAS, 388, 1062
- Contopoulos G., 1956, ZAp, 39, 126
- De Filippis E., Sereno M., Bautz M. W., Longo G., 2005, ApJ, 625, 108
- Debattista V. P., Moore B., Quinn T., Kazantzidis S., Maas R., Mayer L., Read J., Stadel J., 2008, ApJ, 681, 1076
- Dubinski J., 1994, ApJ, 431, 617
- Evans N. W., 1993, MNRAS, 260, 191
- Evans N. W., 1994, MNRAS, 267, 333
- Evrard A. E., Metzler C. A., Navarro J. F., 1996, ApJ, 469, 494
- Fabian A. C., Hu E. M., Cowie L. L., Grindlay J., 1981, ApJ, 248, 47
- Fabjan D., Borgani S., Rasia E., Bonafede A., Dolag K., Murante G., Tornatore L., 2011, MNRAS, 416, 801
- Fabricant D., Rybicki G., Gorenstein P., 1984, ApJ, 286, 186
- Fang T., Humphrey P., Buote D., 2009, ApJ, 691, 1648
- Feng J. L., Kaplinghat M., Tu H., Yu H., 2009, J. Cosmology Astropart. Phys., 7, 4
- Feng J. L., Kaplinghat M., Yu H., 2010, Physical Review Letters, 104, 151301
- Flores R. A., Allgood B., Kravtsov A. V., Primack J. R., Buote D. A., Bullock J. S., 2007, MNRAS, 377, 883
- Fox D. C., Pen U.-L., 2002, ApJ, 574, 38
- Gastaldello F., Buote D. A., Humphrey P. J., Zappacosta L., Bullock J. S., Brighenti F., Mathews W. G., 2007, ApJ, 669, 158
- Gavazzi R., 2005, A&A, 443, 793
- Gottlöber S., Yepes G., 2007, ApJ, 664, 117
- Ho S., Bahcall N., Bode P., 2006, ApJ, 647, 8
- Hopkins P. F., Bahcall N. A., Bode P., 2005, ApJ, 618, 1
- Jing Y. P., Mo H. J., Borner G., Fang L. Z., 1995, MNRAS, 276, 417
- Jing Y. P., Suto Y., 2002, ApJ, 574, 538
- Kassiola A., Kovner I., 1993, ApJ, 417, 450
- Kasun S. F., Evrard A. E., 2005, ApJ, 629, 781
- Kawahara H., 2010, ApJ, 719, 1926
- Kazantzidis S., Kravtsov A. V., Zentner A. R., Allgood B., Nagai D., Moore B., 2004, ApJ, 611, L73
- Krause E., Pierpaoli E., Dolag K., Borgani S., 2011, ArXiv e-prints
- Kravtsov A. V., Vikhlinin A., Nagai D., 2006, ApJ, 650, 128
- Kriss G. A., Cioffi D. F., Canizares C. R., 1983, ApJ, 272, 439
- Lau E. T., Nagai D., Kravtsov A. V., Zentner A. R., 2011, ApJ, 734, 93
- Lee J., 2006, ApJ, 643, 724

- Lee J., Jing Y. P., Suto Y., 2005, *ApJ*, 632, 706
- Lee J., Suto Y., 2004, *ApJ*, 601, 599
- Macciò A. V., Dutton A. A., van den Bosch F. C., 2008, *MNRAS*, 391, 1940
- Mahdavi A., Chang W., 2011, *ApJ*, 735, L4
- Mohr J. J., Evrard A. E., Fabricant D. G., Geller M. J., 1995, *ApJ*, 447, 8
- Morandi A., Limousin M., 2011, *ArXiv e-prints*
- Morandi A., Limousin M., Rephaeli Y., Umetsu K., Barkana R., Broadhurst T., Dahle H., 2011, *MNRAS*, 416, 2567
- Morandi A., Pedersen K., Limousin M., 2010, *ApJ*, 713, 491
- Muñoz-Cuartas J. C., Macciò A. V., Gottlöber S., Dutton A. A., 2011, *MNRAS*, 411, 584
- Navarro J. F., Frenk C. S., White S. D. M., 1997, *ApJ*, 490, 493
- Paz D. J., Lambas D. G., Padilla N., Merchán M., 2006, *MNRAS*, 366, 1503
- Pierre M., Pcaud F., Juin J. B., Melin J. B., Valageas P., Clerc N., Corasaniti P. S., 2011, *MNRAS*, 414, 1732
- Piffaretti R., Jetzer P., Schindler S., 2003, *A&A*, 398, 41
- Planck Collaboration Ade P. A. R., Aghanim N., Arnaud M., Ashdown M., Aumont J., Baccigalupi C., Balbi A., Banday A. J., Barreiro R. B., et al. 2011, *ArXiv e-prints*
- Pratt G. W., Arnaud M., Piffaretti R., Böhringer H., Ponman T. J., Croston J. H., Voit G. M., Borgani S., Bower R. G., 2010, *A&A*, 511, A85
- Puchwein E., Bartelmann M., 2006, *A&A*, 455, 791
- Reblinsky K., 2000, *A&A*, 364, 377
- Rossi G., Sheth R. K., Tormen G., 2011, *MNRAS*, p. 1032
- Sayers J., Golwala S. R., Ameglio S., Pierpaoli E., 2011, *ApJ*, 728, 39
- Schneider P., Ehlers J., Falco E. E., 1992, *Gravitational Lenses*
- Sereno M., 2007, *MNRAS*, 380, 1207
- Sereno M., De Filippis E., Longo G., Bautz M. W., 2006, *ApJ*, 645, 170
- Sereno M., Ettori S., Baldi A., 2011, *ArXiv e-prints*
- Shaw L. D., Holder G. P., Bode P., 2008, *ApJ*, 686, 206
- Shaw L. D., Weller J., Ostriker J. P., Bode P., 2006, *ApJ*, 646, 815
- Splinter R. J., Melott A. L., Linn A. M., Buck C., Tinker J., 1997, *ApJ*, 479, 632
- Springel V., White S. D. M., Hernquist L., 2004, in S. Ryder, D. Pisano, M. Walker, & K. Freeman ed., *Dark Matter in Galaxies Vol. 220 of IAU Symposium, The shapes of simulated dark matter halos.* p. 421
- Stark A. A., 1977, *ApJ*, 213, 368
- Suwa T., Habe A., Yoshikawa K., Okamoto T., 2003, *ApJ*, 588, 7
- Thomas P. A., Colberg J. M., Couchman H. M. P., Efstathiou G. P., Frenk C. S., Jenkins A. R., Nelson A. H., Hutchings R. M., Peacock J. A., Pearce F. R., White S. D. M., 1998, *MNRAS*, 296, 1061
- Vera-Ciro C. A., Sales L. V., Helmi A., Frenk C. S., Navarro J. F., Springel V., Vogelsberger M., White S. D. M., 2011, *MNRAS*, p. 1100
- Vikhlinin A., Murray S., Gilli R., Tozzi P., Paolillo M., Brandt N., Tagliaferri G., Bautz M., Allen S., Donahue M., Evrad A., Flanagan K., Rosati P., Borgani S., Giacconi R., Weisskopf M., Ptak A., et al. 2009, in *astro2010: The Astronomy and Astrophysics Decadal Survey Vol. 2010 of ArXiv Astrophysics e-prints, X-ray Cluster Cosmology.* p. 305
- Wang Y.-G., Fan Z.-H., 2004, *ApJ*, 617, 847
- White M., Hernquist L., Springel V., 2002, *ApJ*, 579, 16
- Zaroubi S., Squires G., Hoffman Y., Silk J., 1998, *ApJ*, 500, L87